

**Gauge Theory of Quantum Gravity**

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# Gauge Theory of Quantum Gravity

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## Abstract

A gauge theory of quantum gravity is formulated, in which an internal, field dependent metric is introduced which non-linearly realizes the gauge fields on the non-compact group  $SL(2, C)$ , while linearly realizing them on  $SU(2)$ . Einstein's  $SL(2, C)$  invariant theory of gravity emerges at low energies, since the extra degrees of freedom associated with the quadratic curvature and the internal metric only dominate at high energies. In a fixed internal metric gauge, only the the  $SU(2)$  gauge symmetry is satisfied, the particle spectrum is identified and the Hamiltonian is shown to be bounded from below. Although Lorentz invariance is broken in this gauge, it is satisfied in general. The theory is quantized in this fixed, broken symmetry gauge as an  $SU(2)$  gauge theory on a lattice with a lattice spacing equal to the Planck length. This produces a unitary and finite theory of quantum gravity.

## 1. Introduction

The problem of quantum gravity continues to be a central issue in modern physics<sup>1</sup> and is considered by many to be the greatest challenge in theoretical physics today. Despite the fact that considerable effort has been devoted by many physicists over a period of 40 years to solve the problem of quantum gravity, no significant success has been achieved in this quest. There have been two views on how to attack the problem: the first assumes that a correct technical solution is required, which will unite gravitation theory with local, relativistically invariant quantum field theory. The second seeks a new and perhaps radical departure from conventional Einstein gravitational theory and the axioms of local, relativistic field theory. In conventional particle theories, it is assumed that the fields propagate on a non-dynamical, fixed background spacetime, while in modern gravity theories the spacetime is curved and is the dynamical field, i.e. the “arena” plays a central dynamical role in the theory. It could perhaps be that new methods specially invented to quantize diffeomorphism invariant theories, like Einstein’s theory of gravitation, will succeed in producing a consistent quantum gravity scheme without the need to change quantum field theory or Einstein’s theory of gravitation. However, it may be also true that some new idea that changes the conventional picture is needed to successfully unite gravity and quantum mechanics.

One of the obviously serious drawbacks to discovering the “right” quantum gravity theory is that there is no body of experimental data to guide us in our search. In the development of early quantum mechanics, there was the Planck theory of blackbody radiation and the photo-electric effect to show the way, and the Michelson-Morely experiment was an important experimental signpost that guided the invention of special relativity. All we have in our quest for a quantum gravity theory is a perception that there must

exist a “beautiful” and mathematically consistent paradigm that will be accepted by the theoretical physics community as the correct quantum gravity theory.

Attempts to solve the problem using perturbation theory with an expansion around a fixed classical background lead either to an unrenormalizable theory or to a violation of unitarity, or both. The problem with unitarity is possibly more severe than the lack of renormalizability, because whereas higher order theories can be found that are renormalizable, they suffer from ghost poles and lack of unitarity<sup>2</sup>. Moreover, whether the theory is renormalizable or not is probably irrelevant, for quantum gravity effects will not become important before the Planck energy  $\sim 10^{19}$  GeV, when the renormalizability and convergence of the perturbation theory will surely break down above the Planck energy. A standard field theoretic treatment, based on perturbation theory using Feynman diagrams obtained from path integrals, or from a canonical formulation, fails for Einstein’s theory at two-loop level and for all loops when matter is included, and the feature of diffeomorphism invariance seems to make a non-perturbative approach necessary. Therefore, it seems imperative that we use non-perturbative methods to quantize gravity.

Another problem is that standard classical Einstein gravity, based on a metric and a connection, does not have the form of a classical Yang-Mills gauge theory. In Einstein’s theory of gravity, the metric is the dynamical field, and the connection is restricted to being a function of the metric by metricity and torsion-free constraints, while in Yang-Mills theory the connection is the dynamical variable and the metric is a constant,  $\delta_{ab}$ . This makes Einstein’s theory appear to be disturbingly different from the Yang-Mills structure of all other modern field theories, in particular, the standard model of elementary particles, which has been remarkably successful in its agreement with experimental data. Thus, it would be desirable to seek a quantum gravity theory that is not only consistent, but is also easy to unify with the successful standard model.

A promising quantization scheme was developed by Ashtekar<sup>3</sup>, in which Einstein's theory of gravity is reformulated in terms of new variables. This program had some appealing features that made it suitable for a canonical quantization of the gravitational field using Dirac's methods of quantization. Emphasis was shifted from the metric to the connection, which made it possible to obtain polynomial-type constraints which could be solved. This appeared to remove the serious technical problems related to the constraints in earlier attempts to quantize canonical formulations of gravity. Recently, Capovilla, Dell and Jacobson<sup>4</sup> have gone even further and developed a theory almost purely in terms of the connection. However, other serious problems arose, for inner products of states in the Hilbert space of physical solutions could not be found and observables could not be obtained to calculate physical quantities. Ashtekar's Hamiltonian is complex and reality conditions have to be imposed to extract a real Einstein gravity theory. In effect, this means that not all the Hamiltonian constraints can be solved. Without a solution to this problem and the metric-signature condition, one cannot calculate any physical quantities in the theory.

Attempts have also been made to develop gauge theories of gravity, path integral quantization formulations in Euclidean space, and numerical calculations in simplicial lattice space gravity. These routes to a theory suffered problems with the inherent non-compact nature of the spacetime group underlying conventional Einstein gravity and its various gauge theory generalizations.

In the following, we shall develop a new quantum gravity theory by introducing a different physical principle for small-distance gravity. Thus, we adopt the position that some new physical insight is needed to understand the short-distance behavior of gravity, i.e. "new physics" at short distances is required to solve quantum gravity. We shall spontaneously violate local Lorentz invariance in a fixed, internal metric gauge, while

maintaining a causal theory. In general, the theory is invariant under Lorentz and diffeomorphism transformations. By taking this apparently radical step, we solve many of the current vexing problems in quantum gravity, without violating well-known experimental tests of classical general relativity. Since we only violate Lorentz invariance in the fixed internal metric gauge, and because there is nothing physically special about this gauge, we can quantize the gravitational theory as an  $SU(2)$  gauge theory on a lattice with the lattice spacing  $a = G^{1/2}$ . In this way we obtain a finite and unitary solution for quantum gravity.

Einstein's classical theory of gravity emerges in a low-energy limit, since a coupling constant and a mass control the quadratic gauge piece and the internal metric contributions to the Lagrangian density, and set a high-energy scale of the order of the Planck energy. Any corrections to Einstein's gravitational theory are down by powers of the inverse Planck mass  $\sim E/M_P$ . Thus, Einstein's theory is treated as an "effective" theory, valid at macroscopic distances and is largely independent of small-distance behavior. This point of view is in accord with the modern treatment of quantum field theory, which accepts non-renormalizable interactions suppressed by inverse powers of the cutoff<sup>5</sup>.

We shall quantize the  $SU(2)$  version of the Yang-Mills theory by using non-perturbative loop representation methods<sup>6</sup>, which can be converted to a lattice gauge theory with a length scale  $a$  equal to the Planck length. Arguments have been put forward<sup>7</sup> which suggest that the loop representation applied to quantum gravity naturally leads to  $a$  being equal to the Planck length. This will pave the way for a finite quantum gravity theory, which in our case will be unitary, as well.

A Higgs-type spontaneous symmetry breaking can also be invoked at Planck temperatures through a first-order phase transition. Spontaneous violation of local Lorentz and diffeomorphism invariance has interesting consequences for the problem of time in

quantum cosmology<sup>8</sup>, early Universe cosmology<sup>9</sup>, as well as for the problem of information loss in black hole evaporation<sup>10</sup>.

## 2. $SL(2, \mathbb{C})$ Spinor Gauge Formalism

We shall begin by reviewing the basic properties of  $SL(2, \mathbb{C})$  gauge theory of gravity based on a spinor formalism<sup>11–13</sup>. We define at each point of the four-dimensional space-time manifold, a complex two-dimensional linear space, called the spinor space. The elements of the spinor space are composed of two-dimensional spinors, namely, two-component complex quantities  $\psi^A$ ,  $A=0,1$ .

Let  $p_a^A$  be a normalized spinor basis, i.e. any pair of spinors (dyad):  $p_a^A = (p_0^A, p_1^A)$ , such that

$$p_a^A \epsilon_{AB} p_b^B = \epsilon_{ab}, \quad p_a^A \epsilon^{ab} p_b^B = \epsilon^{AB}, \quad (2.1)$$

where the dyad indices  $a, b = 0, 1$ . Also,

$$\epsilon_{AB} = \epsilon^{AB} = \epsilon_{A'B'} = \epsilon^{A'B'} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (2.2)$$

where a prime on a suffix denotes the complex conjugate operation. The  $\epsilon_{AB}$  satisfy

$$\epsilon_{AC} \epsilon^{BC} = \delta_A^B. \quad (2.3)$$

Spinor indices are raised and lowered according to the rules:

$$\psi_A = \psi^B \epsilon_{BA}, \quad \psi^A = \epsilon^{AB} \psi_B. \quad (2.4)$$

A spinor  $\psi^A$  can be expanded in terms of the dyad  $p_a^A$ , namely

$$\psi^A = \psi^a p_a^A, \quad (2.5)$$

where the scalars  $\psi^a$  denote the dyad components of  $\psi^A$ .

The spinor  $\psi_A$  satisfies the transformation law:

$$\psi_A = \Lambda_A^B \psi'_B, \quad \psi'^A = \psi^B \Lambda_B^A, \quad (2.6)$$

such that

$$\psi_A \psi^A = \psi'_A \psi'^A, \quad (2.7)$$

is an invariant. Here,  $\Lambda_A^B$  denotes the spinor transformation matrix in the transformation law (an element of the group  $SL(2, C)$ ). The spinors are scalars with respect to the group of general coordinate transformations in the base manifold.

The covariant differentiation operator for spinors, which allows spinors to be compared at different spacetime points, is defined by ( $\psi_{,\mu} = \partial/\partial x^\mu$ ):

$$D_\mu \psi_A = \psi_{A,\mu} - \Omega_{A\mu}^B \psi_B, \quad D_\mu \psi^A = \psi^A_{,\mu} + \Omega_{B\mu}^A \psi^B, \quad (2.8)$$

where  $\Omega_{A\mu}^B$  ( $\mu = 0, 1, 2, 3$ ) are four two-by-two complex matrices, denoting the spinor connections, which are subject to the transformation law:

$$\Omega'^B_{A\mu} = (\Lambda^{-1})^D_A \Omega^C_{D\mu} \Lambda^B_C + (\Lambda^{-1})^C_{A,\mu} \Lambda^B_C. \quad (2.9)$$

The spinor connections are traceless matrices:  $\Omega^A_{A\mu} = 0$ . The metric structure of spacetime and the spinor space can be connected by introducing a Hermitian spinor vector, a set of four Hermitian two-by-two matrices:

$$\sigma^\mu_{AB'} = \bar{\sigma}^\mu_{B'A}, \quad (2.10)$$

which satisfy the orthogonality conditions:

$$\sigma^\mu_{AB'} \sigma^\nu{}^{AB'} = \delta^\mu{}_\nu, \quad \sigma^\mu_{AB'} \sigma^\mu{}^{CD'} = \delta_A{}^C \delta_{B'}{}^{D'}. \quad (2.11)$$

Using these algebraic conditions, we obtain the relation:

$$\sigma_{AC'}^\mu \sigma^{\nu BC'} + \sigma_{AC'}^\nu \sigma^{\mu BC'} = g^{\mu\nu} \delta_A^B, \quad (2.12)$$

and

$$g_{\mu\nu} = \sigma_{\mu AC'} \sigma_{\nu}^{AC'}, \quad (2.13)$$

where  $g_{\mu\nu}$  is the spacetime pseudo-Riemannian metric tensor. In flat spacetime with  $g_{\mu\nu} = \eta_{\mu\nu}$ , where  $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ , we can choose the  $\sigma$ 's as the three Pauli spin matrices and the unit matrix:

$$\begin{aligned} \sigma_{AB'}^0 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \sigma_{AB'}^1 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \sigma_{AB'}^2 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, & \sigma_{AB'}^3 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \quad (2.14)$$

We impose the conditions:

$$D_\nu \sigma_{AB'}^\mu = 0, \quad (2.15)$$

which determine the relation between the spinor connection  $\Omega_\mu$  and the Christoffel connection  $\Gamma_{\mu\nu}^\lambda$ :

$$\Omega_{A\mu}^C = \frac{1}{2} \sigma_\nu^{CB'} (\sigma_{AB'}^\rho \Gamma_{\rho\mu}^\nu + \sigma_{AB',\mu}^\nu). \quad (2.16)$$

We introduce the representation:

$$D_\mu p_a^A = A_{a\mu}^b p_b^A. \quad (2.17)$$

The coefficients  $A_{a\mu}^b$ , which form a set of four  $2 \times 2$  complex matrices, will be taken to be the connections of the gauge theory. They satisfy the traceless condition:  $A_{a\mu}^a = 0$  and the transformation law:

$$A_{c\mu}^d = (\Lambda^{-1})_c^a A_{a\mu}^b \Lambda_b^d - (\Lambda^{-1})_c^a \Lambda_{a,\mu}^d. \quad (2.18)$$

For a normalized spinor basis:  $p_0^A = (1, 0)$ ,  $p_1^A = (0, 1)$ , we have

$$A_{a\mu}^b = \Omega_{A\mu}^C p_a^A p_C^b. \quad (2.19)$$

The gauge field (curvature) is given by

$$G_{a\mu\nu}^b = A_{a\mu,\nu}^b - A_{a\nu,\mu}^b + A_{a\mu}^c A_{c\nu}^b - A_{a\nu}^c A_{c\mu}^b. \quad (2.20)$$

We shall adopt the matrix notation:

$$G_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu} + [A_\mu, A_\nu]. \quad (2.21)$$

The gauge field has the usual transformation law under spinor transformations:

$$G_{a\mu\nu}^b = (\Lambda^{-1})_a^c G_{c\mu\nu}^d \Lambda_d^b. \quad (2.22)$$

We also have that

$$D_\nu D_\mu p_a^A - D_\mu D_\nu p_a^A = G_{a\mu\nu}^b p_b^A \quad (2.23)$$

and the traceless condition:  $G_{A\mu\nu}^A = 0$ . For an arbitrary spinor  $\psi^A$ , we have

$$D_\nu D_\mu \psi^A - D_\mu D_\nu \psi^A = (\Omega_{B\mu,\nu}^A - \Omega_{B\nu,\mu}^A + \Omega_{B\mu}^C \Omega_{C\nu}^A - \Omega_{B\nu}^C \Omega_{C\mu}^A) \psi^B. \quad (2.24)$$

This yields

$$G_{A\mu\nu}^B = (\Omega_{\mu,\nu} - \Omega_{\nu,\mu} + [\Omega_\mu, \Omega_\nu])_A^B. \quad (2.25)$$

The Bianchi identities take the form:

$$\epsilon^{\alpha\beta\gamma\delta} (G_{\beta\gamma,\delta} + [G_{\beta\gamma}, \Omega_\delta])_A^B = 0. \quad (2.26)$$

The Riemann curvature tensor is related to the gauge field by the equation:

$$R^\alpha{}_{\beta\gamma\delta} = -(G_{A\gamma\delta}^C \sigma_{CB'}^\alpha + G_{B'\gamma\delta}^{C'} \sigma_{AC'}^\alpha) \sigma_\beta^{AB'}, \quad (2.27)$$

and

$$G_{A\gamma\delta}^B = -\frac{1}{2}R^\alpha{}_{\beta\gamma\delta}\sigma_\alpha^{BC'}\sigma_{AC'}^\beta, \quad (2.28)$$

where

$$R^\delta{}_{\alpha\beta\gamma} = \Gamma_{\alpha\gamma,\beta}^\delta - \Gamma_{\alpha\beta,\gamma}^\delta + \Gamma_{\alpha\gamma}^\mu\Gamma_{\mu\beta}^\delta - \Gamma_{\alpha\beta}^\mu\Gamma_{\mu\gamma}^\delta, \quad (2.29)$$

is the Riemann curvature tensor.

### 3. The Lagrangian Density

We must now formulate a Lagrangian density which guarantees that the Hamiltonian is bounded from below and that there are no ghost states and violations of unitarity. Since we are trying to construct an  $SL(2, C)$  gauge theory of gravity, we are confronted with the usual problems of negative probabilities and negative energy when using a non-compact group<sup>14,15,8</sup>. This may be avoided by using an internal metric tensor  $s_{ab}$ , which is Hermitian, non-negative and is a matrix of scalar fields<sup>16-20</sup>. This leads us to the Lagrangian density:

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_M, \quad (3.1)$$

where

$$\mathcal{L}_1 = -\frac{1}{\kappa}\sigma Tr(\Sigma^{\mu\nu}G_{\mu\nu}), \quad (3.2)$$

$$\mathcal{L}_2 = -\sigma\left\{\frac{1}{4\alpha}Tr(s^{-1}G^{\dagger\mu\nu}sG_{\mu\nu}) + \frac{m^2}{8}Tr[(s^{-1}\nabla_\mu s)(s^{-1}\nabla^\mu s)]\right\}, \quad (3.3)$$

$$\mathcal{L}_3 = \sigma\left\{\nabla_\mu\phi^\dagger s\nabla^\mu\phi - V(\phi^\dagger s\phi)\right\}, \quad (3.4)$$

where

$$\sigma \equiv \sqrt{-g} = [-\det(\sigma_{\mu ab'}\sigma_\nu^{ab'})]^{1/2}, \quad (3.5)$$

with  $g = \det(g_{\mu\nu})$ , and

$$\Sigma_{a\mu\nu}^b = \frac{1}{2}(\sigma_{\mu ac'}\sigma_{\nu}^{bc'} - \sigma_{\nu ac'}\sigma_{\mu}^{bc'}). \quad (3.6)$$

Moreover,  $\kappa = 8\pi G$ ,  $\alpha$  is a dimensionless coupling constant,  $m$  is a mass and  $\mathcal{L}_M$  is the matter Lagrangian density. We have also included a coupling to two complex scalar fields  $\phi$ , where  $\phi$  transforms as

$$\phi' = \Lambda^\dagger \phi. \quad (3.7)$$

Moreover,  $V$  is a potential which is a function of the  $SL(2, C)$  invariant quantity  $\phi^\dagger s \phi$ .

The covariant derivative of  $s$  with respect to the gauge potential  $A_\mu$  is defined by

$$\nabla_\lambda s = s_{,\lambda} - A_\lambda^\dagger s - s A_\lambda, \quad (3.8)$$

and the internal metric  $s$  transforms as

$$s' = \Lambda^\dagger s \Lambda, \quad (3.9)$$

where  $s^\dagger$  is the Hermitian conjugate of  $s$ .

The Lagrangian density  $\mathcal{L}$  is invariant under  $SL(2, C)$  gauge transformations and, as we shall see in the next section, the Hamiltonian with the correct constraints imposed on it is bounded from below.

By using the anti-commutation relations of the  $\sigma$ 's, we get

$$Tr(\Sigma^{\mu\nu}\Sigma_{\alpha\beta}) = \frac{1}{2}(\delta_\beta^\mu\delta_\alpha^\nu - \delta_\alpha^\mu\delta_\beta^\nu) + \frac{1}{2}i\epsilon^{\mu\nu}{}_{\alpha\beta}. \quad (3.10)$$

Suppressing dyad indices, we can write Eq. (2.30) in the form:

$$G_{\mu\nu} = \frac{1}{2}\Sigma^{\alpha\beta}R_{\mu\nu\alpha\beta}. \quad (3.11)$$

Substituting this expression into (3.2), using (3.10) and the cyclic identity obeyed by the curvature tensor:

$$\epsilon^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} = 0, \quad (3.12)$$

we get the Einstein-Hilbert Lagrangian density:

$$\mathcal{L}_1 = \frac{1}{2\kappa} \sqrt{-g} R, \quad (3.13)$$

where  $R_{\alpha\beta} = R^\sigma{}_{\alpha\sigma\beta}$  and  $R = R^\alpha{}_\alpha$  are the Ricci tensor and the Ricci scalar, respectively. The Lagrangian density (3.3) is quadratic in the Riemann curvature tensor. The most general Lagrangian density involving quadratic curvature invariants will contain 16 curvature invariants— 4 Weyl invariants, 4 Ricci invariants and 8 mixed invariants. Here,

$$\begin{aligned} C_{\alpha\beta\gamma\delta} = & R_{\alpha\beta\gamma\delta} - \frac{1}{2}(g_{\alpha\gamma}R_{\beta\delta} + g_{\beta\delta}R_{\alpha\gamma} - g_{\alpha\delta}R_{\beta\gamma} - g_{\beta\gamma}R_{\alpha\delta}) \\ & + \frac{1}{6}(g_{\alpha\gamma}g_{\beta\delta}R - g_{\alpha\delta}g_{\beta\gamma}R), \end{aligned} \quad (3.14)$$

is the Weyl conformal tensor which satisfies:

$$C_{\beta\alpha\delta}^\alpha = 0. \quad (3.15)$$

We have chosen the form of  $\mathcal{L}_2$  by analogy with Yang-Mills theory.

#### 4. Classical Hamiltonian

The Hamiltonian constraint formalism and canonical formalism for an  $SL(2, C)$  gauge invariant Lagrangian density of the type we propose for  $\mathcal{L}_2$  in (3.3), has been analyzed in detail by Dell, deLyra and Smolin<sup>19</sup> and Popović<sup>20</sup>.

We shall consider the matter-free case,  $T_{\mu\nu} = 0$ , and we shall also choose  $\phi_\mu = 0$ . The Lagrangian becomes:

$$\mathcal{L}_G = \mathcal{L}_1 + \mathcal{L}_2. \quad (4.1)$$

The Lagrangian density (4.1) is independent of  $\det(s)$  and we can therefore choose  $\det(s) = 1$ . Also,  $s$  has no singularities and we are able to choose a gauge transformation  $q$  such that  $s = I$  everywhere:

$$s' = q^\dagger s q = I, \quad (4.2)$$

where  $I$  is the unit matrix. Since  $s$  is a Hermitian matrix with unit determinant, the condition (4.2) fixes the  $SL(2, C)/SU(2)$  part of the gauge freedom, and one is left only with the  $SU(2)$  gauge invariance. In the gauge (4.2), the particle content of the theory becomes manifest. The  $SU(2)$  gauge field is coupled to a multiplet of massive spin-one vector bosons in the adjoint representation, with the couplings chosen so that the theory can be extended by means of the metric  $s$  to be invariant under the larger group  $SL(2, C)$ , which is isomorphic to the homogeneous Lorentz group  $SO(3, 1)$ . Thus, the Hermitian metric field  $s$  has conspired with the  $SL(2, C)/SU(2)$  gauge field to produce massive vector fields, while the gauge fields associated with the compact generators of  $SU(2)$  remain massless. Although local Lorentz invariance is broken in the gauge (4.2), there is nothing physically special about this gauge and in general Lorentz invariance is satisfied. Thus, the Lorentz symmetry breaking is a special representation of the physical theory, and the vacuum state is not broken in any gauge, including the one determined by (4.2). We can therefore choose to quantize the theory in this gauge and not violate any physical principles.

The coupling constant  $\alpha$  and the mass  $m$  are chosen so that at low energies the Lagrangian density  $\mathcal{L}_1$  dominates and yields  $SL(2, C)$  and diffeomorphism invariant Einstein gravity, i.e., the extra degrees of freedom associated with the quadratic pieces and the internal metric  $s$  dominate at high energies. At energies of the order of the Planck mass, where quantum gravity becomes important, the Lagrangian density in the gauge  $s = I$  is

broken down to an  $SU(2)$  invariant gauge theory, which approximates classical Einstein gravity at low energies.

The traceless  $2 \times 2$  complex matrices  $X$  which describe the  $SL(2, C)$  generators can be decomposed into anti-Hermitian and Hermitian pieces with respect to  $s$ :

$$X = i\tau + \lambda, \quad (4.3)$$

where

$$i\tau = \frac{1}{2}(X - s^{-1}X^\dagger s), \quad \lambda = \frac{1}{2}(X + s^{-1}X^\dagger s). \quad (4.4)$$

We have

$$(i\tau) = -s^{-1}(i\tau)^\dagger s, \quad \lambda = s^{-1}\lambda^\dagger s. \quad (4.5)$$

The anti-Hermitian piece  $i\tau$  generates the  $SU(2)$  maximally compact subgroup of  $SL(2, C)$ , which preserves  $s$ , and which is isomorphic to the three-dimensional rotation group  $SO(3)$ . Under an infinitesimal  $\lambda$  transformation:

$$\delta s = s\lambda + \lambda^\dagger s. \quad (4.6)$$

The  $SL(2, C)$  gauge potential  $A_\mu$  can be decomposed into anti-Hermitian and Hermitian pieces with respect to  $s$ <sup>19</sup>:

$$A_\mu = iV_\mu + B_\mu, \quad (4.7)$$

where

$$iV_\mu = \frac{1}{2}(A_\mu - s^{-1}A_\mu^\dagger s), \quad B_\mu = \frac{1}{2}(A_\mu + s^{-1}A_\mu^\dagger s). \quad (4.8)$$

This decomposition is not gauge invariant, for the two pieces mix under gauge transformations.

The  $V_\mu$  fields correspond to the familiar  $SU(2)$  Yang-Mills gauge fields, while the  $B_\mu$  fields are related to the  $SL(2, C)/SU(2)$  part of the gauge group. We have

$$\nabla_\mu s = s_{,\mu} - 2sB_\mu. \quad (4.9)$$

In general, the Lagrangian density (4.1) is invariant under the full  $SL(2, C)$  group of transformations.

By substituting (4.7) into (3.2), we obtain

$$\mathcal{L}_1 = -\frac{1}{\kappa}\sqrt{-g}[Tr(L^{\mu\nu}U_{\mu\nu} + S^{\mu\nu}C_{\mu\nu})], \quad (4.10)$$

where

$$U_{\mu\nu} = V_{\mu,\nu} - V_{\nu,\mu} + [V_\mu, B_\nu] + [B_\mu, V_\nu], \quad (4.11)$$

$$C_{\mu\nu} = b_{\mu\nu} - [V_\mu, V_\nu], \quad (4.12)$$

and

$$b_{\mu\nu} = B_{\mu,\nu} - B_{\nu,\mu} + [B_\mu, B_\nu]. \quad (4.13)$$

Here, we have used the fact that, in view of (3.13), the Lagrangian density  $\mathcal{L}_1$  is real. This leads to the condition:

$$Tr(L^{\mu\nu}D_{\mu\nu} + S^{\mu\nu}U_{\mu\nu}) = 0, \quad (4.14)$$

where

$$D_{\mu\nu} = b_{\mu\nu} + V_{\nu,\mu} - V_{\mu,\nu}. \quad (4.15)$$

If we set  $s = I$ , then the Lagrangian density  $\mathcal{L}_2$ , which contains the quadratic Yang-Mills piece, becomes:

$$\mathcal{L}_2 = -\sqrt{-g}\left[\frac{1}{4\alpha}Tr(K^{\mu\nu}K_{\mu\nu} + W^{\mu\nu}W_{\mu\nu}) + \frac{m^2}{2}Tr(B^\mu B_\mu)\right], \quad (4.16)$$

where

$$K_{\mu\nu} = f_{\mu\nu} + [B_\mu, B_\nu], \quad W_{\mu\nu} = \tilde{\nabla}_\mu B_\nu - \tilde{\nabla}_\nu B_\mu. \quad (4.17)$$

Here, we have

$$f_{\mu\nu} = V_{\nu,\mu} - V_{\mu,\nu} + [V_\mu, V_\nu], \quad \tilde{\nabla}_\mu B_\nu = B_{\nu,\mu} + [V_\mu, B_\nu]. \quad (4.18)$$

The spontaneous symmetry breaking that occurs here, is the kind associated with a non-linear realization of the fields on the noncompact group  $SL(2, C)$ , inducing a linear realization on the maximal compact subgroup  $SU(2)^{16-20}$ . This differs from the standard type of Higgs breaking in that the vacuum expectation values:  $\langle \phi \rangle_0 = \langle A_\mu \rangle_0 = \langle \Omega \rangle_0 = 0$ , while for the *internal* metric  $\langle s \rangle_0 = s_0$ , we have a non-vanishing vev. There are no scalar particles left over after the longitudinal parts of the massive vector fields have been accounted for, and there are no self-interacting pieces for  $s$  in the Lagrangian density, since they are forbidden by the  $SL(2, C)$  gauge invariance. In particular, the ground state is invariant under the full group of  $SL(2, C)$  gauge transformations.

The group  $SL(2, C)$  is the double covering group of the homogeneous Lorentz group  $SO(3, 1)$  and has the Lie algebra:

$$Y_m = \sigma_m/2, \quad Z_m = i\sigma_m/2, \quad m = 1, 2, 3, \quad (4.19)$$

$$[Y_m, Y_n] = i\epsilon_{mnk}Y^k, \quad [Y_m, Z_n] = i\epsilon_{mnk}Z^k, \quad [Z_m, Z_n] = -i\epsilon_{mnk}Y^k, \quad (4.20)$$

where the  $\sigma_m$  are the three Pauli spin matrices. Let us use the three-vector notation:

$$V_m B^m = V \cdot B, \quad \epsilon_{mnk} V^n B^k = (V \times B)_m. \quad (4.21)$$

Then, the Lagrangian densities  $\mathcal{L}_1$  and  $\mathcal{L}_2$  can be written:

$$\mathcal{L}_1 = -\frac{1}{\kappa} \sqrt{-g} (L^{\mu\nu} \cdot U_{\mu\nu} + S^{\mu\nu} \cdot C_{\mu\nu}), \quad (4.22)$$

and

$$\mathcal{L}_2 = -\sqrt{-g} \left[ \frac{1}{4\alpha} (K^{\mu\nu} \cdot K_{\mu\nu} + W^{\mu\nu} \cdot W_{\mu\nu}) + \frac{m^2}{2} B_\mu \cdot B^\mu \right], \quad (4.23)$$

where

$$U_{\mu\nu} = V_{\mu,\nu} - V_{\nu,\mu} + V_\mu \times B_\nu + B_\mu \times V_\nu, \quad (4.24)$$

$$C_{\mu\nu} = b_{\mu\nu} + V_\mu \times V_\nu, \quad (4.25)$$

$$b_{\mu\nu} = B_{\mu,\nu} - B_{\nu,\mu} - B_\mu \times B_\nu. \quad (4.26)$$

Moreover, we have

$$K_{\mu\nu} = V_{\nu,\mu} - V_{\mu,\nu} - V_\mu \times V_\nu + B_\mu \times B_\nu, \quad (4.27)$$

$$W_{\mu\nu} = \tilde{\nabla}_\mu B_\nu - \tilde{\nabla}_\nu B_\mu, \quad (4.28)$$

$$\tilde{\nabla}_\mu B_\nu = B_{\nu,\mu} - V_\mu \times B_\nu. \quad (4.29)$$

In the work of Popović<sup>20</sup> and Dell, de Lyra and Smolin<sup>19</sup>, a set of constraint equations was derived using Dirac Hamiltonian constraint analysis<sup>21</sup>. In a local geodesic (Fermi) frame of coordinates, we have  $g_{\mu\nu} = \eta_{\mu\nu}$ , and the canonical Hamiltonian for  $\mathcal{L}_2$  is defined by

$$H_2 = \pi_m^i \dot{V}_i^m + P_m^i \dot{B}_i^m - \mathcal{L}_2, \quad (4.30)$$

where

$$\pi_m^0 = \frac{\partial \mathcal{L}_2}{\partial \dot{V}_0^m} = 0, \quad P_m^0 = \frac{\partial \mathcal{L}_2}{\partial \dot{B}_0^m} = 0, \quad (4.31)$$

$$\pi_m^i = \frac{\partial \mathcal{L}_2}{\partial \dot{V}_i^m} = -\frac{1}{\alpha} K_m^{0i}, \quad (4.32)$$

$$P_m^i = \frac{\partial \mathcal{L}_2}{\partial \dot{B}_i^m} = -\frac{1}{\alpha} W_m^{0i}. \quad (4.33)$$

The primary constraints are given by (4.31), while the secondary constraints are of the form:

$$\{\pi_0^m, H_2\} = C^m = \pi_{m,i}^i + \epsilon^{mnk} \pi_n^i V_i^k + \epsilon^{mnk} P_n^i B_i^k \approx 0, \quad (4.34)$$

$$\{P_0^m, H_2\} = D^m = P_{m,i}^i - \epsilon^{mnk} P_k^i V_i^n - \epsilon^{mnk} \pi_n^i B_i^k + m^2 B_0^m \approx 0. \quad (4.35)$$

The  $C^m$  generate the usual  $SU(2)$  gauge transformations, while the  $D^m$  correspond to the  $SL(2, C)/SU(2)$  gauge symmetry of the original theory.

When the Dirac brackets are calculated, we find that all second-class constraints become strong inequalities. In particular, it follows that

$$\tilde{C}_m = 0, \quad \pi_m^0 = 0, \quad (4.36)$$

are first-class constraints, where

$$\tilde{C}_m \equiv C^m - \epsilon^{mnk} B_0^n P_0^k, \quad (4.37)$$

while

$$D_m = 0, \quad P_m^0 = 0, \quad (4.38)$$

are second-class constraints.

In the presence of the constraints, we obtain the total Hamiltonian:

$$H = H_1 + H_2, \quad (4.39)$$

where

$$H_2 = \frac{\alpha}{2} [(\pi_m^i)^2 + (P_m^i)^2] + \frac{m^2}{2} [(B_i^m)^2 + (B_0^i)^2] + \frac{1}{4\alpha} [(K_{ij}^m)^2 + (W_{ij}^m)^2] \geq 0, \quad (4.40)$$

and  $H_1$  is the constrained Hamiltonian obtained from the Lagrangian density (4.22). The total Hamiltonian  $H$  is positive and bounded from below and defines a consistent Hamiltonian system. A potential problem with the mass term  $B^\mu B_\mu$ , due to the indefiniteness of the Minkowski metric  $\eta_{\mu\nu}$ , is avoided, because the constraints switch the sign of the term  $m^2 B_0^2$  in the Hamiltonian  $H_2$ . Dell, de Lyra and Smolin<sup>19</sup> have shown in a gauge independent analysis that  $H_2$  is indeed ghost free and tachyonic free and that the boundedness

of the Hamiltonian  $H_2$  from below is a general result in the  $SL(2, C)$  (Lorentz) invariant theory.

#### 4. Non-perturbative Lattice Quantization

The experience of the last thirty or more years of attempts to construct a consistent quantum gravity theory have shown that a perturbative expansion around a fixed classical background spacetime cannot succeed in describing the quantum behavior of the gravitational field. Thus, we have to resort to a non-perturbative approach to quantize the gravitational field. The perturbative field theory based on the Lagrangian density (3.1) does not lead to a renormalizable field theory<sup>19</sup>. Possible additional dimension four contributions can be added to (3.1), which make the perturbative theory renormalizable, but at the cost of violating unitarity<sup>22</sup>. At any rate, even if the theory was perturbatively renormalizable, the perturbation expansion and the renormalizability will be expected to break down above the Planck energy, and render the whole scheme useless. One possible approach to constructing a non-perturbative theory is to use a loop representation method<sup>6</sup> to quantize the gravitational gauge field theory, described by the Lagrangian density. We shall use this method in conjunction with the Wilson-type lattice gauge approach<sup>23,24</sup>, in which both space and time are discretized in the Euclidean version of the theory with a lattice spacing  $a$ , which we choose to be equal to the Planck length,  $a = G^{1/2}$ .

The lattice site will be denoted by a four-vector  $n$ . The four-dimensional integration will be replaced by a sum:

$$\int d^4x \rightarrow a^4 \Sigma_n. \quad (4.41)$$

We introduce the non-integrable phase factor of a wavefunction  $\psi(x)$ :

$$U(x, x') = \exp \left\{ i \int_C Y \cdot V_\mu dy^\mu \right\}, \quad (4.42)$$

associated with the gauge potential  $V_\mu$ . Thus, for the gauge function:

$$\Phi(\theta) = \exp[iY \cdot \theta(x)], \quad (4.43)$$

we have

$$\psi(x) \rightarrow \Phi(\theta)\psi(x), \quad (4.44)$$

$$\bar{\psi}(x) \rightarrow \bar{\psi}(x)\Phi^\dagger(\theta)$$

and

$$U'(x, x') = \Phi(\theta)U(x, x')\Phi^\dagger(\theta). \quad (4.45)$$

The lattice versions of these gauge transformations are:

$$\psi_n \rightarrow \Phi_n\psi_n, \quad (4.46)$$

$$\bar{\psi}_n \rightarrow \bar{\psi}_n\Phi_n^\dagger, \quad (4.47)$$

and

$$U(n + \hat{\mu}, n)' = \Phi_{n+\hat{\mu}}U(n + \hat{\mu}, n)\Phi_n^\dagger, \quad (4.48)$$

where

$$\Phi_n = \exp(iY\theta_n). \quad (4.49)$$

Here,  $\hat{\mu}$  is a four-vector of length  $a$  in the direction of  $\mu$ .

The action for our lattice gauge gravity theory is given by

$$S_G = \Sigma_p (-g_n)^{1/2} \left\{ -\frac{1}{\kappa} (L^{n,\mu\nu} \cdot U_{n,\mu\nu} + S^{n,\mu\nu} \cdot C_{n,\mu\nu}) + \frac{1}{2\alpha} \{Tr[\exp(ia^4 K_{n,\mu\nu})] \right. \\ \left. + Tr[\exp(ia^4 W_{n,\mu\nu})] \} + \frac{m^2}{2} Tr[\exp(ia^4 B_{n\mu})] \right\}, \quad (4.50)$$

where the sum is over all the plaquettes of the lattice. Moreover, we have

$$\Sigma_{n,\mu\nu} = iL_{n,\mu\nu} + S_{n,\mu\nu}, \quad (4.51)$$

where

$$\Sigma_{an,\mu\nu}^b = \frac{1}{2}(\sigma_{n,\mu ac'}\sigma_{n,\nu}^{bc'} - \sigma_{n,\nu ac'}\sigma_{n,\mu}^{bc'}). \quad (4.52)$$

Also, we have

$$K_{n,\mu\nu} = \partial_\mu V_{n\nu} - \partial_\nu V_{n\mu} - V_{n\mu} \times V_{n\nu} + B_{n\mu} \times B_{n\nu}, \quad (4.53)$$

$$W_{n,\mu\nu} = \partial_\mu B_{n\nu} - \partial_\nu B_{n\mu} + V_{n\nu} \times B_{n\mu} - V_{n\mu} \times B_{n\nu}, \quad (4.54)$$

where

$$\partial_\mu V_{n\nu} = \frac{1}{a}(V_{(n+\hat{\mu})\nu} - V_{n\nu}) \quad (4.55)$$

and  $g_n$  and  $V_{n\mu}$  denote the spacetime metric tensor and the gauge potential field at the site  $n$ . We can recover the continuum limit from expressions of the form:

$$\frac{1}{2}\Sigma_p \left\{ 1 - \frac{a^4}{2} K^{\mu\nu} \cdot K_{\mu\nu} + \dots \right\} \rightarrow -\frac{1}{4} \int d^4x K^{\mu\nu} \cdot K_{\mu\nu}, \quad (4.56)$$

where we have used the tracelessness of the  $SU(2)$  matrices and  $Tr(Y^a Y^b) = \frac{1}{2}\delta^{ab}$ .

We are now in a position to compute the functional generator:

$$Z = \int d[V_{n\mu}, g_n, B_{n\mu}] \mathcal{M}(V_{n\mu}, g_n, B_{n\mu}) \exp(iS_G), \quad (4.57)$$

where  $\mathcal{M}$  is a lattice measure. An important question to investigate is whether there exists a finite lattice formulation of the theory with  $a = G^{1/2}$ , since a continuum limit for a non-renormalizable theory is not necessarily guaranteed by the existence of a fixed point in the  $\beta$  function and a second-order phase transition.

## 5. Concluding Remarks

We have constructed an  $SL(2, C)$  invariant theory of quantum gravity by using a covariant spinor formalism and introducing an  $SL(2, C)$  Lagrangian density, which contains

quadratic Yang-Mills-type contributions. Problems with unitarity are avoided by invoking a non-linear realization of the internal metric  $s$  on the non-compact group  $SL(2, C)$ , which is then realized linearly on  $SU(2)$ . By transforming to the gauge  $s = I$ , the theory is spontaneously broken down to an  $SU(2)$  invariant theory and quantized on a lattice with the lattice spacing equal to the Planck length. Although Lorentz invariance is broken in the gauge  $s = I$ , this happens only in this particular gauge and, in general, the theory is fully Lorentz and diffeomorphism invariant. In the “unitary” gauge  $s = I$ , the Hamiltonian is shown to be bounded from below. A general gauge independent analysis will maintain the positivity of the Hamiltonian as a general feature of the theory. Einstein’s locally Lorentz invariant and diffeomorphism invariant theory will dominate at low energies and produce the standard agreement with observational tests, because the extra degrees of freedom related to the breaking of local Lorentz invariance in the gauge  $s = I$  will be undetectable at low energies. Thus, in this scheme, gravity in the gauge  $s = I$ , becomes an  $SU(2)$  invariant gauge theory, which is dominated by Einstein’s  $SL(2, C)$  invariant gravity theory at large distances.

We then quantize the  $SU(2)$  (or  $SO(3)$ ) invariant theory on a lattice by constructing the discretized action, which then produces a functional generator  $Z$  in terms of the discretized path integral.

Further work must be done to investigate the possible existence of a continuum limit or, alternatively, the existence of a finite renormalizable version of the  $SU(2)$  gauge invariant gravity theory using the finite lattice spacing  $a = G^{1/2}$ . This program is facilitated by the fact that an  $SU(2)$  lattice calculation is not difficult to perform using, for example, the heat kernel technique and Monte Carlo simulation methods<sup>24</sup>.

In contrast to the work of Ashtekar<sup>3</sup>, in which a complex connection is introduced based on a complex  $SO(3, 1; C)$  group structure, or an  $SL(2, C)$  connection which is self-dual, the present theory leads to a real Hamiltonian, and there is no problem in defining physical state vectors and inner products for a Hilbert space.

If we assume that a first-order phase transition occurs at high temperatures,  $T \sim T_c$ , due to a non-vanishing vacuum expectation value:

$$\langle \phi^\dagger s \phi \rangle_0 = v^2, \quad (4.58)$$

corresponding to a minimum in the potential:

$$V'(\phi^\dagger s \phi) = 0, \quad (4.59)$$

then the physical true vacuum will be spontaneously broken, due to the breaking of local and diffeomorphism invariance. This will correspond to a standard Higgs breaking of the true vacuum, which leads to interesting consequences for the problem of time in quantum gravity<sup>8</sup>, early Universe cosmology<sup>9</sup> and black hole evaporation and information loss phenomena<sup>10</sup>. The matter part of the Higgs mechanism associated with the matter fields  $\phi$  will possess a physical particle spectrum free of ghost poles and tachyons, because of the non-linear realization of the  $\phi$  matter section of the Lagrangian density (3.4) through the Hermitian internal metric  $s$ .

Finally, we should note that the present theory could have been formulated in terms of real vierbeins  $e_\mu^a$ , defined by the spacetime metric:

$$g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}. \quad (4.60)$$

A Lagrangian density could be constructed which is invariant under  $SO(3, 1)$  or  $SL(4, R)$  gauge transformations. Then unitarity for the scheme will be guaranteed by non-linearly

realizing these non-compact groups on a real positive internal metric,  $s_{ab}$ . In the particular gauge  $s = I$ , local Lorentz invariance will be broken down to  $SO(3)$  for the  $SO(3, 1)$  theory and  $O(4)$  for the  $SL(4, R)$  theory. As in the  $SL(2, C)$  gauge theory, the Lagrangian density will be invariant under the full non-compact gauge group through the extended internal metric theory.

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